A Comparison of Some Finite Element Time Domain Formulations in Electromagnetics

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Abstract — **This paper compares three full-wave Finite Element Time Domain (FETD) formulations. The first is based on the vector wave equation; the others on Maxwell's equations, viz. the EBHD formulation that discretises $\vec{E}$, $\vec{B}$, $\vec{H}$ and $\vec{D}$ and the EB formulation that discretises only $\vec{E}$ and $\vec{B}$. The latter two formulations use a combination of 1-form and 2-form discretisation to avoid an auxilary mesh. A method for making the EBHD formulation operational is presented and conditions for Finite Difference Time Domain (FDTD)-like explicit operation are discussed. Numerical results for a three dimensional cavity and a coaxial transmission line show that the EBHD formulation has serious performance limitations.**

1 Introduction

For solving the transient response of electromagnetic systems, finite element time domain (FETD) methods using unstructured meshes are suitable for complex geometries; the structured meshes used by finite difference time domain (FDTD) methods limit the achievable accuracy on such geometry. Most FETD methods require matrix solution at each time-step while FDTD is fully explicit; baring errors in geometric modeling FDTD will usually be more computationally efficient than FETD.

Vector full-wave Finite Element Time Domain (FETD) methods commonly fall into two categories; those based on the vector wave equation, and those based on the coupled Maxwell's equations. To date wave equation formulations have been the most popular, and are well described in the common engineering literature, e.g. [1]. This may partly be due to the simplicity of implementing it in basic form given an existing frequency domain full-wave FEM code.

Excluding formulations that call for auxilary meshes, two formulations based on the coupled Maxwell's equations have been proposed. A formulation based on discretising the $\vec{E}$ and $\vec{H}$ fields as well as the $\vec{D}$ and $\vec{B}$ flux densities is presented in [2] and expanded upon in [3]. The other, discretises only the $\vec{E}$ field and $\vec{B}$ flux density [4, 5]. They are respectively called the EBHD formulation and the EB formulation in this paper. The complementarity features of 1-form and 2-from Whitney forms [4] obviates a complementary mesh.

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Using the Newmark-β time-stepping scheme [2] the wave equation formulation can be unconditionally stable, a great advantage when meshes with varying element sizes are desired. Both Maxwell's formulations involve only first order time derivatives, potentially simplifying the implementation of PML mesh termination and dispersive material modeling. The EBHD formulation is claimed to support explicit operation in FDTD style operation, while the EB formulation is simpler than the EBHD formulation.

This paper compares the popular wave equation based formulation and the two coupled Maxwell's formulations. Relative accuracies are compared and the possibility of explicit operation discussed.

2 Formulation

Full-wave FEM formulations are generally used to solve Maxwell's equations in a domain Ω:

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{2}$$

Taking the curl on both sides of (1) and eliminating $\vec{H}$ using (2), the Helmholtz vector wave equation is obtained:

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{E} + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{\partial \vec{J}}{\partial t}. \tag{3}$$

Zero initial values and boundary conditions are specified, making (1, 2) or (3) well posed initial value problems. For simplicity, Direchlet boundary conditions are considered here. More sophisticated mesh termination schemes may be used depending on the nature of the problem being solved, see e.g. [1].

2.1 Field Discretisation

Using the language of Differential Forms [6], the $\vec{E}$ and $\vec{H}$ field quantities are 1-forms that, assuming continuous material parameter variation within elements, fundamentally require tangential inter-element continuity; similarly, the 2-form $\vec{D}$ and $\vec{B}$

flux densities require normal inter-element continuity. These tangential and normal continuity requirements are often equivalently stated as, respectively, curl- and div-conformance. Suitable function spaces for the definition of discrete 1- and 2-form representations are presented in [4]. The concrete hierarchical basis functions of arbitrary order for tetrahedrons defined in [7] and [8] are used for respectively 1-form and 2-form discretisation.

Throughout the notation $\vec{w^{(1)}}$ and $\vec{w^{(2)}}$ will be used for respectively 1-form and 2-form basis functions. E.g. $\vec{w}_{ei}^{(1)}$ refers to the ith 1-form basis function used to discretise an $\vec{E}$ field, and $\vec{w}_{bj}^{(2)}$ refers to the jth 2-form basis function used to discretise a $\vec{B}$ flux density. An $\vec{E}$ field discretised by k basis functions is represented as $\vec{E} = \sum_{i=1}^{k} e_i \vec{w}_{ei}^{(1)}$ where e_i is the degree of freedom (DOF) associated with the ith basis function. Putting the DOFs and basis functions in two column vectors $\{e\}$, $\{w_e^{(1)}\}$, we can write $\vec{E} = \{e\}^T \cdot \{w_e^{(1)}\}$.

Only one differential operator can be applied to each p-form and is unambiguously called the exterior derivative. For 1- and 2-forms this operator is respectively curl and divergence. The curl of 1-forms is a subset of 2-forms. A discrete 2-form can exactly represent the curl of a 1-form. With a discrete 1-form $\vec{A}$ and a discrete 2-form $\vec{B}$ on the same mesh, the operation $\vec{B} = \nabla \times \vec{A}$ has the discrete equivalent $\{b\} = [C_a]\{a\}$; $[C_a]$ is a highly sparse circulation matrix with only 1 or -1 entries [5].

2.2 Vector Wave Equation Formulation

The vector wave equation formulation is well known and is completely discussed in [1, §12]. Using suitable 1-form basis functions the Galerkin procedure is applied to (3). Assuming homogeneous Dirichlet boundary conditions, the semi-discrete differential equation system

$$[M]\frac{d^2\{e\}}{dt^2} + [S]\{e\} + \{f\} = \{0\} \qquad (4)$$

is obtained, where $[M]$ and $[S]$ are the square mass and stiffness matrices, $\{e\}$ the vector of $\vec{E}$ field degrees of freedom (DOFs) and $\{f\}$ the current driving vector. In [1] the calculation of the matrix entries are shown; note that $[M]$ is equivalent to Jin's $[T]$ matrix.

Time integration is performed using the Newmark-β method. See [1, §12] for original references on the Newmark methods. For $\beta \geq \frac{1}{4}$ this method is unconditionally stable.

2.3 EBHD Maxwell's Formulation

The Maxwell's equations (1, 2) are used to implement a leapfrog scheme [3]. The fields $\vec{E}$, $\vec{H}$ are discretised as 1-forms and the flux densities $\vec{D}$, $\vec{B}$ as 2-forms. Writing time as $t = n\Delta t$ where n is the integer time-step and Δt is the constant time step size, we have $\vec{E}(n)$ and $\vec{D}(n)$ and half a time step apart $\vec{H}(n+\frac{1}{2})$ and $\vec{B}(n+\frac{1}{2})$. Starting with known fields and fluxes at $t = n\Delta t$,

$$\begin{aligned} \{b\}^{n+\frac{1}{2}} &= \{b\}^{n-\frac{1}{2}} - \Delta t[C_e]\{e\}^n \\ \{h\}^{n+\frac{1}{2}} &= [\star_b]\{b\}^{n+\frac{1}{2}} && (5) \\ \{d\}^{n+1} &= \{d\}^n + \Delta t[C_h]\{h\}^{n+\frac{1}{2}} - \{j\}^{n+\frac{1}{2}} \\ \{e\}^{n+1} &= [\star_d]\{d\}^{n+1}. && (6) \end{aligned}$$

The $[\star_b]$ and $[\star_d]$ matrices are respectively the magnetic and electric discrete Hodge star operators. The Hodge operator here transforms a 2-form representation into an equivalent 1-form representation [6]. This is how the constitutive relations, i.e. $\vec{H} = \frac{1}{\mu}\vec{B}$ and $\vec{E} = \frac{1}{\epsilon}\vec{D}$, are applied.

A discrete Hodge operator can be constructed by a Galerkin process where both sides of (5) are tested using the $\{w_h^{(1)}\}$ basis functions, resulting in

$$[M_h]\{h\}^{n+\frac{1}{2}} = [P_{bh}]\{b\}^{n+\frac{1}{2}}$$

giving

$$[\star_b] = [M_h]^{-1}[P_{bh}].$$

Here $[M_h]$ is the $\{w_h^{(1)}\}$ mass matrix, and $[P_{bh}]$ the projection of $\{w_b^{(2)}\}$ onto $\{w_h^{(1)}\}$. The matrix entries are

$$\begin{aligned} M_{h_{ij}} &= \int_\Omega \vec{w}_{hi}^{(1)} \cdot \vec{w}_{hj}^{(1)} d\Omega \\ P_{bh_{ij}} &= \int_\Omega \frac{1}{\mu} \vec{w}_{hi}^{(1)} \cdot \vec{w}_{bj}^{(2)} d\Omega. \end{aligned}$$

A discrete Hodge operator for (6) can be derived similarly.

The above Galerkin method is not explicit since the $[M_h]$ and $[M_e]$ mass matrices have to be inverted. In [3] an explicit collocation-based Hodge operator is outlined, but it requires the intergration of the undefined tangential component of 2-form basis functions at interelement boundaries. In light of the EBHD formulation's poor performance using the Galerkin based Hodge operator (see Section 3) the collocation-based operator was not pursued.

2.4 EB Maxwell's Formulation

This formulation also based on (1, 2), discretises $\vec{E}$ as a 1-form and $\vec{B}$ as a 2-form and was first outlined in [4], with practical results in [5]. Because the $\vec{E}$ discretisation is curl-conforming (1) is trivially verified in discrete form:

$$\frac{d}{dt}\{b\} = [C_e]\{e\}.$$

More care is needed for (2). First re-write it as

$$\epsilon\frac{\partial \vec{E}}{\partial t} = \nabla \times (\frac{1}{\mu}\vec{B}) - \vec{J}. \qquad (7)$$

Now a Galerkin procedure is applied by testing both sides of (7) with $\{\vec{w}_e^{(1)}\}$. A Green's identity is used to transfer the curl of $\vec{B}$ to the testing functions, since the 2-form discretisation is not curl-conforming. The discreet counterpart to (7), assuming Dirichlet boundary conditions, is

$$[M_e]\frac{d}{dt}\{e\} = [P_{b,curl(e)}]\{b\} - \{f\}.$$

$[M_e]$ is the $\{w_e^{(1)}\}$ mass matrix, $[P_{b,curl(e)}]$ the projection of $\{w_b^{(2)}\}$ onto $\nabla \times \{w_e^{(1)}\}$ and $\{f\}$ is the projection of $\vec{J}$ onto $\{w_e^{(1)}\}$. The matrix and vector entries are

$$\begin{aligned} M_{e_{ij}} &= \int_\Omega \epsilon \vec{w}_{ei}^{(1)} \cdot \vec{w}_{ej}^{(1)} d\Omega \\ P_{b,curl(e)_{ij}} &= \int_\Omega \frac{1}{\mu} \vec{w}_{ei}^{(1)} \cdot \vec{w}_{bj}^{(2)} d\Omega \\ f_i &= \int_\Omega \vec{J} \cdot \vec{w}_{ei}^{(1)} d\Omega. \end{aligned}$$

$\vec{E}$ and $\vec{B}$ are discretised half a time-step apart as in Section 2.3. Starting with know $\vec{E}$ and $\vec{B}$ at $t = n\Delta t$, the update equations are

$$\{b\}^{n+\frac{1}{2}} = \{b\}^{n-\frac{1}{2}} - \Delta t[C_e]\{e\}^n$$
$$\{e\}^{n+1} = \{e\}^n + [M_e]^{-1}\Delta t([P_{b,curl(e)}]\{b\}^{n+\frac{1}{2}} - \{f\}^{n+\frac{1}{2}}).$$

2.5 Explicit Operation

Explicit operation requires explicit time integration *and* field modeling that results in a diagonal mass matrix. The leapfrog time integration used by the EBDH and EB formulations is explicit, as is the Newmark-β with $\beta = 0$. Explicit time integration will at best result in conditional stability; suitable implicit schemes can achieve unconditional stability.

With rectangular brick elements the mass matrix can be diagonalised by using trapezoidal geometric integration yielding the classic Yee FDTD with the EB or vector wave formulations with $\beta = 0$ [9]. With a non-diagonal mass matrix, $\beta = 0$ results in a better conditioned matrix equation since the $[S]$ matrix is excluded from the LHS.

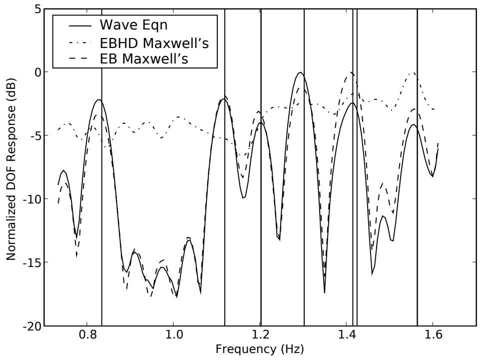

Figure 1: Frequency Response in 1x0.75x0.5m PEC Cavity. Vertical bars represent analytical mode cutoff frequencies.

3 Results

3.1 Cavity Resonance

A PEC cavity has well known resonant modes; the analytical solution is straightforward and can be found in many engineering electromagnetics texts. If a cavity is fed with a wide band pulse designed to excite all the modes, resonant peaks should appear at the modes' cutoff frequencies. This serves as a basic test for the correct functioning of a formulation. In this and the follwing results, the speed of light is normalized to 1 m/s.

A 1x0.75x0.5 m cavity is discretised with a nominal edge length of $\frac{1}{9}$ m using the mixed first-order 1- and 2-form basis functions. A random selection of element edges is excited with a Gaussian pulse modulated by a sine-wave, designed with a center frequency (F_c) of 1.25 Hz and 90% of F_c bandwidth. The systems are run for 4096 time-steps with $\Delta t = 0.005$s. The average frequency response of a random selection of $\vec{E}$-edge DOFs is measured and is shown in Fig. 1. The wave equation and EB formulations show the expected resonant peaks while the EBHD formulation shows no clear peaks. Further investigation using higher order elements (mixed second order for 1-forms and fully linear 2-forms) showed no improvement in the EBHD results, as did geometrical refinement.

3.2 Eigen solution

The source-free semi-discrete forms of the formulations are transformed to the frequency domain by replacing $\frac{d}{dt}$ by $-j\omega$ and all the field/flux unknowns except for $\vec{E}$ are eliminated. The remaining equation can be cast in the standard form of a generalized eigen problem. Solving this eigen problem on a cavity yields the cavity mode cutoff wavenumbers as eigenvalues. Replacing $\frac{d}{dt}$ by $-j\omega$ implies perfect time integration, hence any error in the eigen solution is determined solely by errors in geometric field interpolation. In other words, it provides an upper bound for the accuracy of a given scheme using perfect time integration.

The vector wave equation formulation yields the well known eigen system,

$$[S]\{e\} = \omega^2 [M]\{e\}.$$

Using the Galerkin discrete Hodge operator for the EBHD formulation, the eigen system is

$$[P_{de}][C_h][M_h]^{-1}[P_{bh}][C_e]\{e\} = \omega^2 [M_e]\{e\}.$$

For the EB formulation it is

$$[P_{b,curl(e)}][C_e]\{e\} = \omega^2 [M_e]\{e\}.$$

The respective eigen systems were solved using the same mesh as in Section 3.1. The results are shown in Table 1.

Formulation	Mode Cutoff Wavenumber				
Analytical	5.24	7.02	7.55	7.55	8.18
Wave Eqn.	5.23	7.01	7.52	7.54	8.12
EBHD	1.17	1.24	1.28	1.31	1.36
EB	5.23	7.01	7.52	7.54	8.12

Table 1: Analytic vs. Computed mode cutoff frequencies

Mirroring previous results, the EBHD spectrum has many spurious modes. Around the expected wavenumbers the EBHD spectrum also reveals regularly spaced spurious modes. Again, neither higher order bases, nor mesh refinement, improved the EBHD result; the other formulations converged at the expected rate [7]. The Wave equation and EB formulations are equivalent for the eigen problem since they both model exactly the same spaces.

3.3 Coaxial TEM Mode Dispersion

In [10] a waveguide result shows reasonable performance using the EBHD formulation. A similar experiment is done using the TEM mode of a coaxial waveguide. The TEM mode has no dispersion, making interpretation of the results simple. A 10 m length of Coaxial guide with inner and outer radii of respectively $\frac{2}{3}$ m and 1 m was simulated using a mesh with 0.3 m nominal edge length. The TEM mode was launched at $z = 0$ using a differentiated Gaussian time waveform with a center frequency of 0.5 Hz. The response at $z = 5$ m was recorded; the simulation was timestepped for 14 s, ensuring no reflections from the short at $z = 10$ m have reached $z = 5$ m. The phase of the transfer function from $z = 0$ to $z = 5$ m is calculated using the FFT. The wavenumber is extracted by dividing out the 5 m signal travel path. The result is shown in Fig. 2.

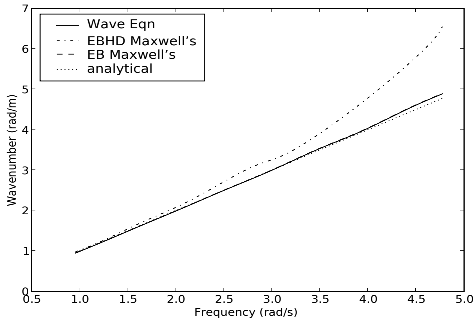

Figure 2: Calculated vs. Analytical Dispersion Relation on Coaxial Waveguide

4 Conclusion

Three FETD formulations were presented, and a procedure for making the EBHD formulation operational was discussed, as were the conditions neccesary for explicit operation. Numerical results for a cavity and coaxial waveguide were presented. It was seen that the vector wave and EB formulations perform similarly, while the EBHD formulation suffers from spurious modes in the cavity and suffers from excessive dispersion at higher frequencies in the coaxial waveguide.

References

[1] J. Jin, *The Finite Element Method in Electromagnetics*, 2nd ed. John Wiley and Sons, 2002.

[2] J.-F. Lee and Z. Sacks, "Whitney elements time domain (WETD) methods," *IEEE Trans. Magn.*, vol. 31, no. 3, pp. 1325–1329, May 1995.

[3] J.-F. Lee, R. Lee, and A. Cangellaris, "Time-domain finite-element methods," *IEEE Trans. Antennas Propagat.*, vol. 45, no. 3, pp. 430–442, Mar. 1997.

[4] J. C. Nedelec, "Mixed finite elements in $\Re^3$," *Numerische Mathematik*, vol. 35, pp. 315–341, 1980.

[5] M.-F. Wong, O. Picon, and V. Fouad Hanna, "A finite element method based on Whitney forms to solve Maxwell equations in the time domain," *IEEE Trans. Magn.*, vol. 31, no. 3, pp. 1618–1621, May 1995.

[6] G. A. Deschamps, "Electromagnetics and differential forms," *Proc. IEEE*, vol. 69, no. 6, pp. 676–969, June 1981.

[7] J. P. Webb, "Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements," *IEEE Trans. Antennas Propagat.*, vol. 47, no. 8, pp. 1244–1253, Aug. 1999.

[8] M. M. Botha, "Fully hierarchical divergence-conforming basis functions on tetrahedral cells, with applications," *International Journal for Numerical Methods in Engineering (in press)*, 2006, available: www.interscience.wiley.com. DOI: 10.1002/nme.1906.

[9] T. Rylander and A. Bondeson, "Stability of explicit-implicit hybrid time-stepping schemes for Maxwell's equations," *Journal of Computational Physics*, vol. 179, no. 2, pp. 426–438, July 2002.

[10] T. V. Yioultsis, N. V. Kantartzis, C. S. Antonopoulos, and T. D. Tsiboukis, "A fully explicit Whitney element - time domain scheme with higher order vector finite elements for three-dimensional high-frequency problems," *IEEE Trans. Magn.*, vol. 34, no. 5, pp. 3288–3291, Sept. 1998.