Conforming Arbitrary Order Hexahedral/Tetrahedral Hybrid Discretisation

Neilen Marais and David B. Davidson

Abstract

A new method for constructing $H(curl)$ or $H(div)$ conforming hexahedral/tetrahedral hybrid meshes of arbitrary discretisation order is presented. Tetrahedral elements that conform directly to hexahedral elements are constructed, avoiding the need for pyramidal or other joining element types. The effectiveness of this method is investigated in a finite element framework, and some caveats pertaining to higher order pyramidal elements are raised.

Introduction

Numerical solution of full-wave or eddy current electromagnetic problems often call for $H(curl)$ or $H(div)$ conforming discreet representation. Conforming representations have been defined on many element shapes. Tetrahedrons and hexahedrons represent two largely complementary shapes. Unstructured tetrahedrons excel at modeling complex geometries; structured Cartesian hexahedrons (i.e. bricks) facilitate the efficient representation of regular geometries. Hybrid meshes utilising mixed 1st order conforming pyramidal elements [1, 2] to connect the tetrahedral and hexahedral regions have been presented for hybrid FEM/FDTD PDE methods [3] and integral methods [4]. When dealing with electrically large problems or if very high accuracy is required, the use of higher order field representation is desirable [5]. While higher order pyramidal repre-

sentations of up to second [1] and mixed third [2] order have been published, they seem to suffer from spurious modes as shown below. In this Letter a different conforming mesh scheme is presented; suitably constrained higher order tetrahedral elements are used to represent the hexahedral degrees of freedom (DOFs) at the hybrid boundary. Numerical validation of this scheme shows that spurious modes are avoided and that higher order convergence is achieved. Furthermore, the avoidance of a third element class (i.e. pyramids) leads to simpler program construction.

Constructing hexahedral-conforming tetrahedra

Mixed p'th order Cartesian product $H(curl)$ hexahedral(hex) basis functions take the form

$$\vec{w}^{\text{hex}}_{(u)} = P^p_v(v)P^p_w(w)P^{p-1}_u(u)\hat{u}, \tag{1}$$

where u, v, w are the element-local coordinates, $\hat{u}, \hat{v}, \hat{w}$ the covariant component basis vectors and P^p is a p'th order univariate polynomial. Similar definitions hold for $\vec{w}^{\text{hex}}_{(v)}$ and $\vec{w}^{\text{hex}}_{(w)}$.

Each hexahedral face on the interface between the hexahedral and tetrahedral meshes (Γ) is connected connected to two tetrahedrons as shown in Fig.

For $H(curl)$ discretisation, basis functions must be tangentially continuous on each hexahedral face. Using standard tetrahedral $H(curl)$ bases complete to the same order as the hexahedral base, this is not possible since the tetrahedral basis functions relating to face and non-diagonal edge DOFs will be identically zero on one of the two triangular faces.

On any hexahedral face the normally varying coordinate is constant, hence the mixed p'th order basis functions described in (1) will contain multivariate polynomial terms of at most order $2p-1$. Hence, tetrahedral $H(curl)$ elements complete to order $2p-1$ that conform to the hex-face geometry (as in Fig. 1) can tangentially match the hexahedral basis functions exactly. A new tetrahedral basis that conforms to the hexahedral basis is constructed from linear

combinations of the orginal order $2p-1$ basis. Conformance requires

$$\hat{n} \times \vec{w}_i^{\text{hex}} = \hat{n} \times \vec{w}_i^{\text{hct}} \text{ on } \Gamma, \tag{2}$$

where $\hat{n}$ is the joining face normal and $\vec{w}_i^{\text{hct}}$ a hexahedral-conforming tetrahedral basis function.

For mixed 1st order discretisation (DOFs as shown in Fig. 1), tetrahedral edge functions $\vec{w}_{u1}^{\text{tet}}, \vec{w}_{u2}^{\text{tet}}, \vec{w}_{v1}^{\text{tet}}, \vec{w}_{v2}^{\text{tet}}, \vec{w}_{d1}^{\text{tet}}$ of the form $\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i$ and $\vec{w}_{d2}^{\text{tet}}$ of the form $\zeta_i \nabla \zeta_j + \zeta_j \nabla \zeta_i$ are needed. Applying (2) and solving for $\vec{w}_{u1}^{\text{hct}}$,

$$\vec{w}_{u1}^{\text{hct}} = \vec{w}_{u1}^{\text{tet}} + 0.5\vec{w}_{d1}^{\text{tet}} + 0.5\vec{w}_{d2}^{\text{tet}}. \tag{3}$$

Similar expressions result for $\vec{w}_{u2}^{\text{hct}}, \vec{w}_{v1}^{\text{hct}}$ and $\vec{w}_{v2}^{\text{hct}}$ with only the sign of the $\vec{w}_{d2}^{\text{tet}}$ term changing if the edge is connected to the ending node of the diagonal. A similar procedure using a normal continuity condition in place of (2) and $H(div)$ bases can be used to construct an $H(div)$ conforming hybrid mesh.

Implementation to arbitrary order

While possible, it would be intractable to construct the hexahedral-conforming tetrahedral bases analytically for high order discretisation. A numerically exact solution is obtained by solving a local projection problem on each hexahedral hybrid boundary face Γ_K. Given the mixed p'th order hexahedral basis on Γ_K, $\mathbf{w}^{\text{hex}} = [\vec{w}_1^{\text{hex}}, \vec{w}_2^{\text{hex}}, \ldots, \vec{w}_{n_{\text{hex}}}^{\text{hex}}]$, and the basis of order $2p-1$ on the connected tetrahedral faces $\mathbf{w}^{\text{tet}} = [\vec{w}_1^{\text{tet}}, \vec{w}_2^{\text{tet}}, \ldots, \vec{w}_{n_{\text{tet}}}^{\text{tet}}]$, we want to find a local $n_{\text{hex}} \times n_{\text{tet}}$ transform matrix ${}^{\text{hc}}\mathbf{T}_K$ such that $\mathbf{w}^{\text{hct}} = {}^{\text{hc}}\mathbf{T}_K \mathbf{w}^{\text{tet}}$ where $\mathbf{w}^{\text{hct}}$ is the local hexahedral-conforming tetrahedral basis. Defining the inner product $< \vec{a}, \vec{b} >_{\Gamma_K} = \int_{\Gamma_K} (\hat{n} \times \vec{a}) \cdot (\hat{n} \times \vec{b}) dA$ and the matrices $\mathbf{M}_K$ and $\mathbf{P}_K$ with entries $M_{Kij} =< \vec{w}_j^{\text{tet}}, \vec{w}_i^{\text{tet}} >$ and $P_{Kij} =< \vec{w}_j^{\text{tet}}, \vec{w}_i^{\text{hex}} >$, we solve

$${}^{\text{hc}}\mathbf{T}_K = \mathbf{P}_K \mathbf{M}_K^{-1}. \tag{4}$$

Used with arbitrary order tetrahedral [5] and hexahedral [6] basis functions, straightforward implementation of arbitrary order hybrid meshes is possible.

The local ${}^{\mathrm{hc}}\mathbf{T}_K$ matrices are assembled into a global $n_{\mathrm{tot}} \times n_{\mathrm{tet}}$ transform matrix ${}^{\mathrm{hc}}\mathbf{T}$, where n_{tot} is the total number of DOFs and n_{tet} the number of tetrahedral DOFs; order $2p-1$ bases are used on the hybrid boundary and mixed p'th order throughout the rest of the tetrahedral mesh. Hybrid system matrices can be constructed as $\mathbf{A}_{\mathrm{hyb}} = \mathbf{A}_{\mathrm{hex}} + {}^{\mathrm{hc}}\mathbf{T}\,\mathbf{A}_{\mathrm{tet}}{}^{\mathrm{hc}}\mathbf{T}^t$, where $\mathbf{A}_{\mathrm{hex}}$ has been zero extended to $n_{\mathrm{tot}} \times n_{\mathrm{tot}}$.

Numerical Results

To test the proposed hybrid mesh, the eigen-solution of the vector Helmholtz equation,

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0, \tag{5}$$

in a 19x23x29 m PEC cavity (speed of light normalised to 1 m/s) is obtained, yielding the cavity mode wavenumbers. Numerical results using hexahedral, hybrid hexahedral-tetrahedral and two different sets of pyramidal elements are compared in Table 1. The hexahedral mesh has a cell-size of 29/4 m. The pyramidal mesh is formed by splitting each hex-element into six pyramids. The hybrid mesh utilises half of the hexahedral mesh; the other half is meshed with unstructured tetrahedrons that conform to the hex faces on Γ. The hexahedral elements apply Gauss-Lobatto mass lumping [6], making them suitable for explicit time-domain FEM methods. Both pyramidal element sets are identical at mixed 1st order and show no spurious modes. For mixed 2nd order, the Coulomb elements [1] ("Pyramids 1" in Table 1) suffer from spurious modes throughout the spectrum, while the Graglia [2] pyramids exhibit a limited number of spurious modes at frequencies ranging from about 1/50 to 1/4 of the lowest physical cavity mode eigenvalue. In Table 1, "Pyramids 2" show the first four spurious eigenvalues, while in "Pyramids 3" the spurious values have been removed. The hybrid mesh does not suffer from spurious modes and delivers

accurate results.

The convergence of the hexahedral and hybrid solution eigenvalues as the total number of DOFs increases is compared in Fig. 2. The rate of convergence is unaffected by the use of the hybrid mesh, while the accuracy per degree of freedom of the pure hexahedral and hybrid meshes are similar.

Conclusion

A new approach for constructing $H(curl)$ and $H(div)$ conforming hexahedral / tetrahedral hybrid meshes that offers straightforward implementation of arbitrary order hybrid discretisations has been proposed. The correct functioning of $H(curl)$ hybrids has been demonstrated using the finite element method eigen-solution of a rectangular cavity. The higher order hybrid solution errors converge at the expected rate while being free of spurious modes. Compared to existing hybrids using pyramidal elements, the complication of an additional element type is removed, while avoiding the problem of spurious modes that currently available higher order pyramidal elements have been found to suffer from. This new approach paves the way for higher order adaptations of existing [3, 4] hybrid mesh methods.

References

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Table Captions

Table 1. First four eigenvalues for a rectangular cavity

Figure Captions

Figure 1. Interface between between hexahedral and tetrahedral mesh with degrees of freedom required for mixed 1st order hybrid shown.

Figure 2. Convergence of hexahedral and hybrid mesh eigenvalues for mixed order 1 through 3 discretisation.

Solution type	Mode eigen-value k_0^2 $(\text{rad.s}^{-1})^2$				RMS error %	total DOFs
	I	II	III	IV		
Analytic	0.030393	0.039075	0.045997	0.057732	—	—
Mixed 1st order discrete						
Hexahedra	0.028862	0.036075	0.042648	0.053793	6.78	75
Pyramids	0.030821	0.040152	0.046792	0.058448	1.88	459
Hybrid	0.029659	0.037784	0.043875	0.055175	3.80	234
Mixed 2nd order discrete						
Hexahedra	0.030384	0.039048	0.045968	0.057700	0.056	854
Pyramids 1	0.007825	0.007877	0.007970	0.010426	—	2774
Pyramids 2	0.000619	0.000679	0.000683	0.000779	—	3062
Pyramids 3	0.030402	0.039100	0.046025	0.057765	0.055	3062
Hybrid	0.030394	0.039078	0.046002	0.057767	0.031	1596

Table 1:

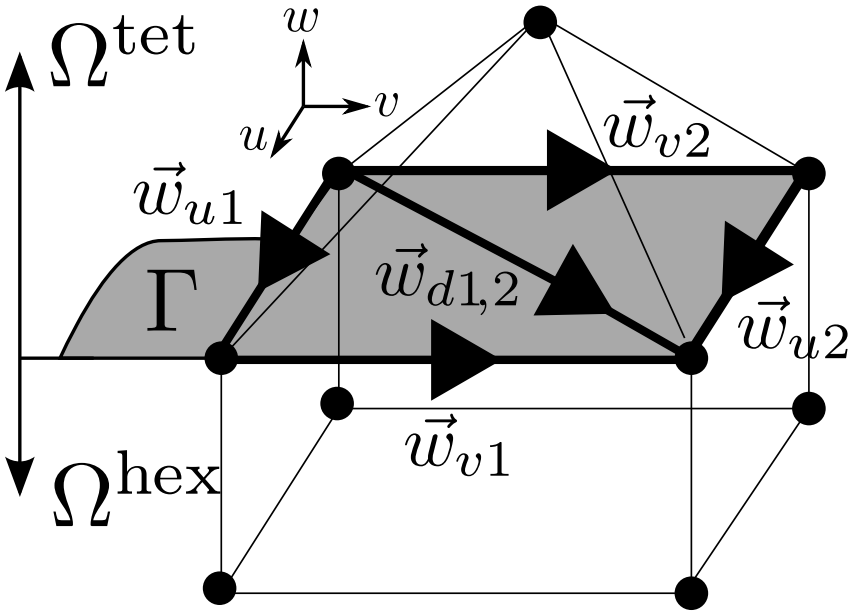
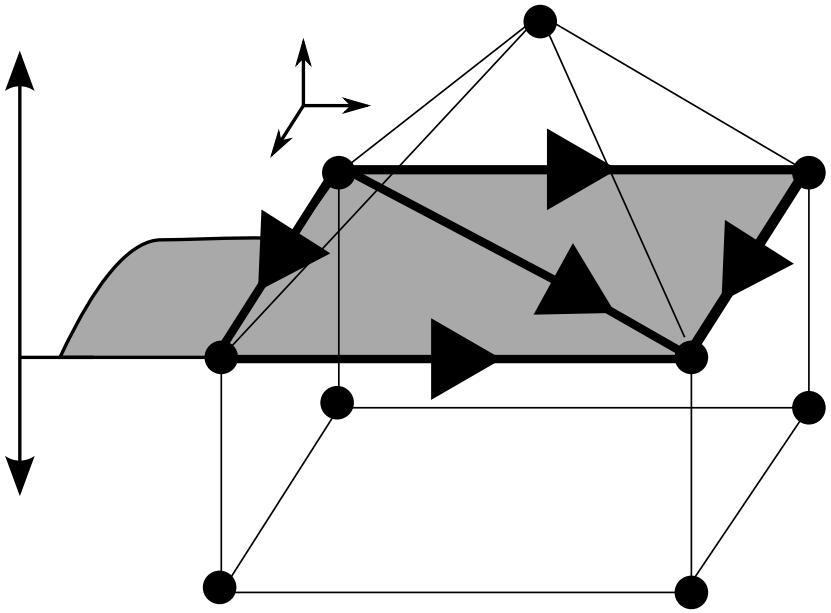

Ω^{tet}
w
v
u
$\vec{w}_{v2}$
$\vec{w}_{u1}$
Γ
$\vec{w}_{d1,2}$
$\vec{w}_{u2}$
$\vec{w}_{v1}$
Ω^{hex}

Figure 1:



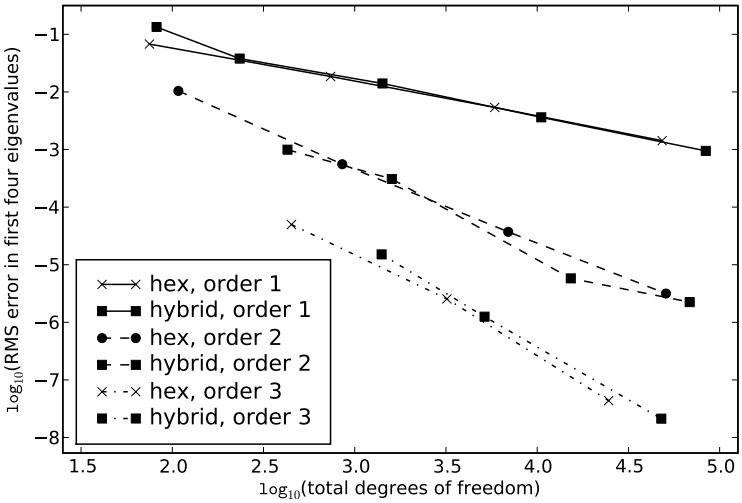

−1
−2
−3
−4
−5
−6
−7
−8
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
log10(RMS error in first four eigenvalues)
log10(total degrees of freedom)
hex, order 1
hybrid, order 1
hex, order 2
hybrid, order 2
hex, order 3
hybrid, order 3

Figure 2: